April 1997

PAR–LPTHE 97-13
cond-mat/9704221

Weak Randomness for large q-State Potts models in Two Dimensions

M. Picco
*LPTHE**
Université Pierre et Marie Curie, PARIS VI
Université Denis Diderot, PARIS VII
Boite 126, Tour 16, 1er étage
4 place Jussieu
F-75252 Paris CEDEX 05, FRANCE

ABSTRACT

We have studied the effect of weak randomness on q-state Potts models for $q > 4$ by measuring the central charges of these models using transfer matrix methods. We obtain a set of new values for the central charges and then show that some of these values are related to one another by a factorization law.

PACS numbers: 64.60.Fr,05.70.Jk,75.40.Mg

*Laboratoire associé No. 280 au CNRS

The effect of weak randomness on second order phase transitions has been the subject of many recent works. These studies were either analytical [1, 2, 3] or numerical [4, 5, 6]. Recently, models with first order phase transitions have also been investigated. According to general arguments, it is expected that a model with a first order phase transition will behave like a second order phase transition in presence of weak disorder [7, 8]. This has been put on a more rigorous level in [9] and checked numerically in [10] where the 8-state Potts model was studied in presence of disorder. In this work, it was also found that the critical exponents of the second order phase transitions are the ones of the Ising model. In a recent study, Cardy considered the case of N coupled Ising models with weak disorder [11]. By considering on an equal footing the coupling between the N Ising models and the disorder, Cardy was able to show that this model flows to N decoupled Ising models. This study was extended to the case of N coupled 3-state Potts models with disorder, with similar conclusions [12].

Based on these results, it was suggested by Cardy that *all* 2-d models will behave like Ising models in presence of weak disorder. The purpose of this Letter is to check the validity of this conjecture. More precisely, if we assume that a model with first order phase transition will behave like an Ising model in presence of disorder, to how many Ising models does it correspond ? An easy way to answer this question, suggested by Cardy, is to measure the central charge of the model in presence of disorder [11]. Then, if this model behaves like M decoupled Ising models, the central charge will simply be $C = M \times \frac{1}{2}$.

In this Letter we present a study in which we measured the central charge for some large q-state Potts model, in presence of weak disorder ($q = 5, 8, 10, 12$ and 256). It *does not* have the central charge of an integer number of Ising models in general. This is the first result. The second result is that for $q = 2^N$, the central charge is $c_{q=2^N} = N \times \frac{1}{2}$. This, in particular, includes the case studied in [10] ($q = 8$) for which an Ising-like behavior was obtained. Our third result is a factorization law. For a general q, we found that if $q = q_1 \times q_2$, then $c_q = c_{q_1} + c_{q_2}$.

Simulations were performed by computing the free energy on long strips with varying width L. It is well known how to relate the free energy to the central charge in the periodic case [13, 14]

$$f_L = f_\infty + \frac{\pi c}{6L^2} + \cdots \tag{1}$$

The free energy being negative, we denote here by f_L the free energy times (-1), *i.e.*

$$f_L = \frac{1}{L} \lim_{N\to\infty} \frac{1}{N} \log Z_{L,N} \ . \tag{2}$$

N is the length of the strip and $Z_{L,N}$ the partition function. Thus this gives us a very fast way of computing the central charge, the only problem being to compute the free energy on such strips. This can be done by iterating the transfer matrix.

We used two different algorithms to perform our measurements. First an algorithm which iterates the transfer matrix, using a sparse-matrix factorization with periodic boundary conditions. The advantage of using sparse-matrix factorization is that it reduces the number of elementary operations from q^{2L} to Lq^L per iterated row [15]. This algorithm is quite efficient for small values of q but for larger values we used another

algorithm. In this second algorithm, we used the fact that not all the elements of the transfer matrix are different. Then, starting with equally probable distributed states, we can only keep the values which are different in our iteration. The only problem is to construct the different elements of the transfer matrix. This becomes particularly complicated because of the presence of the disorder. For instance, for $L = 6$, the number of different elements would be 203. (The number $N_p(L)$ of such elements for a fixed width L is given by

$$N_p(L) = \sum_{i_2=1}^{2} \sum_{i_3=1}^{m_3} \sum_{i_4=1}^{m_4} \cdots \sum_{i_L=1}^{m_{L-1}} 1 \tag{3}$$

with $m_i = max(i_2, i_3, \cdots, m_{i-1}) + 1$.) Then we have to compute the recursion relations between these 203 elements. This leads us to compute $203 \times 203 \times 64$ polynomials (the last 64 are for the number of possible configurations of disorder). The advantage of this algorithm compared to the sparse-matrix one is that it does not depend on the values of q. Then, we can simulate with the same cost in time the free energy for any q-state Potts model. There is still a limit due to the fast increase of the number of polynomials. For $L = 7$, there would be $877 \times 877 \times 128$ such polynomials. Thus, with this algorithm, we were only able to perform simulations with a lot of statistics up to $L = 6$.

The disorder is simulated by allowing the bonds J to take two values, J_0 and J_1, with equal probability. The critical temperature is determined by solving the equation [16]

$$\frac{1 - e^{-\beta J_0}}{1 + (q-1)e^{-\beta J_0}} = e^{-\beta J_1} \ . \tag{4}$$

The strength of the disorder which was chosen is such that $J_0/J_1 = 10$. The reasons for such a choice are as follows. It is well known that if we add disorder to a model with a second order phase transition, then the strength of the disorder will be related to a crossover length. This crossover length can be seen like an average distance between impurities. The stronger the disorder, the smaller the distance between these impurities. In order to detect the effect of the disorder, this average distance should be smaller than the lattice size that we used. In the case of simulations that we report here, we want this crossover length to be as close as possible to the lattice spacing. A direct estimate of this crossover length is too difficult to perform. So we have first made simulations for only one value of q ($q = 8$) for different values of the disorder J_0/J_1 and measured $f(2) - f(3)$. For a small disorder (up to $J_0/J_1 \simeq 5$), this value does not change much from the pure case. Then, we notice a brutal change between $5 < J_0/J_1 < 20$. A similar result was also obtained for $q = 5$. A second constraint is given by the importance of the fluctuations. The stronger the disorder, the stronger the fluctuations of the measured values of the free energy will be. The compromise that we made was to take the value $J_0/J_1 = 10$.

Our simulations have been performed for the following values of q : $q = 5, 8, 10, 12$ and 256. For each of these value of q, we have computed the free energy for $L = 2, 3, 4, 5$ and 6. For $L = 2, 3$ and 4 the transfer matrix was iterated 10^9 times, for $L = 5$ we made 4.10^8 iterations and for $L = 6$, 2.10^7 iterations. Errors were computed by separating the date in $\simeq 1000$ independent runs (for $L = 2, 3, 4$ we made 1000 runs of 10^6 iterations, *etc.*) and then we took the average over these independent runs and the error bars were computed by root-mean-square average of deviations. The values of the free energies

that we measured are reported in Table 1 with the error in the last quoted digits in parentheses. Some simulations were also performed on larger strip widths ($L = 7$ and 8) for $q = 5$ but only over $\simeq 10^6$ iterations. Because we were not able to accumulate enough data for these larger strip widths, we will not report about them here, but let us just mention that they give consistent results. We also performed simulations for other values of q ($q = 6, 11, 16, 24, 64$) but with low statistics. Again, the results for these values of q will be used to check the consistency of our results.

The next step is to compute the central charge. This was done by performing a fit with eq. (1). Then c is obtained by a two-point fit and is given by

$$c_1(q, L) = \frac{6}{\pi} \frac{L^2(L+1)^2}{2L+1} (f_L(q) - f_{L+1}(q)) \; . \tag{5}$$

The values that we obtain with such a fit are reported in Table 2. We see that the values of the central charges c_1 change with increasing L, as can be expected with such small L and thus we cannot obtain reliable values for the central charge without increasing the strips width. At this point, it is important to notice that

$$\Delta c_1(q, L) \simeq \frac{3}{\pi} L^3 (\Delta f_L(q) + \Delta f_{L+1}(q)) \tag{6}$$

for large L. Thus, for some fixed $\Delta c_1(q, L)/c_1(q, L)$, we need to measure $f_L(q)$ such that $\Delta f_L(q) \simeq L^{-3}$. But, as explained above, the time to iterate the transfer matrix increases very quickly with L and the free energies measured are only very weakly self-averaging. Thus the best strategy is certainly not to obtain $c(q)$ as the limit of $c_1(q, L)$ for large L. A better strategy would be to add corrections to eq. (1). Blöte and Nightingale, in an extensive study of the q-state Potts models, showed that the fit is improved with a correction of the following form [17]

$$f_L = f_\infty + \frac{\pi c}{6L^2} + \frac{b}{L^4} + \cdots \tag{7}$$

and this only for $q \leq 4$, *i.e.* for second order phase transition. (In [17], $a = \frac{c\pi}{6}$ was some constant not yet associated with the central charge.) For $q > 4$, such a fit is completely inconsistent. In Table 3, we report the results obtained from a three-point fit of $f(q, L)$ with eq. (7). To show the relevance of such a fit for small L, we also report the values obtained from the free energy computed by Blöte and Nightingale for $q = 2, 3$ and 4. First, let us look at the results for $q = 2, 3$ and 4 (without disorder). For these q's, we see that the deviation of c_2 from the real values ($c(2) = 0.5$, $c(3) = 0.8$ and $c(4) = 1$) is of order $\Delta c/c \simeq 1/100$ already for a three-point fit with $L = 3 - 5$ or $L = 4 - 6$. The original motivation of our study was to check if $c(q)$ could take half-integer values. For this purpose, a precision of $1/100$ is more than enough. Now looking at the results for $c_2(q, L)$ obtained in a three-point fit with $L = 4 - 6$ for $q > 4$, we see that the errors due to the fluctuations are already larger than what is expected from an extrapolation of the results for $q = 2, 3$ and 4. Thus the values obtained by this three-point fit should be very close of the real result. By averaging the results from $L = 3 - 5$ and $L = 4 - 6$, we obtain $c_2(5) = 1.15 - 1.2, c_2(8) = 1.45 - 1.55, c_2(10) = 1.6 - 1.7, c_2(12) = 1.8 - 1.85$ and $c_2(256) = 4 - 4.1$

We also have to mention that other type of corrections to eq. (1) should be incorporated. It has been shown by Cardy that $\log(L)$ corrections exist, due to the presence of irrelevant operators [18]. However, these corrections are very small and should not be important at our level of precision.

Let us now discuss our results. First, for $q = 8$, we know that under the influence of the disorder, the model will behave like an Ising model. We obtain $c \simeq 1.5$ which means that the $q = 8$ state Potts model, in presence of disorder, behaves like three decoupled Ising models. More generally, a $q = 2^N$ state Potts model will have a central charge $c(q) = N.\frac{1}{2} = Nc(2)$. We have checked this result by computing the central charge for $q = 256$, $c(256) \simeq 4 = 8c(2)$ but also for some over values ($q = 16, 64$) for which we have some small data that we will not report here but which is compatible with this rule (we measured $c(16) \simeq 2 \pm 0.1$ and $c(64) \simeq 3 \pm 0.1$). In addition we have $c(4) = 2c(2)$, thus the result $c(q = 2^N) = Nc(2)$ is valid for any N. Then, for $q = 5$, we obtained a value which is very far from any half-integer, $c(5) = 1.15 - 1.2$. This would presumably correspond to a new conformal field theory. The next result is for $q = 10$. For $q = 10$, we obtained $c(10) = 1.6 - 1.7$. Thus we found $c(10) = c(5) + c(2)$ (up to some large errors). Again, we do not know of any conformal field theory with such a central charge. (However, a similar value for c has been obtained for the frustrated XY model ($c = 1.66 \pm 0.04$) [19].) Thus we observe the factorization law

$$c(q = q_1 \times q_2) = c(q_1) + c(q_2) \; . \tag{8}$$

In order to check this factorization law, we have also computed the central charge for $q = 12$. There we obtained $c(12) \simeq 1.8 = 2c(2) + c(3)$ which corroborate our factorization law. This was also checked for some other values of q but with smaller data. All the other cases that we simulated give completely compatible results with this factorization law (we obtained $c(6) \simeq 1.3 \pm 0.1 \simeq c(2) + c(3)$ and $c(24) \simeq 2.3 \pm 0.1 \simeq 3c(2) + c(3)$).

Then, a further consistency test is to perform a simulation with a different boundary condition on the strip. If we choose free boundary conditions, then eq. (1) is replaced by

$$f_L = f_\infty + \frac{f_0}{L} + \frac{c\pi}{24L^2} + \cdots \tag{9}$$

We have an additional term ($\frac{f_0}{L}$) which corresponds to the surface free energy. The computation of $c(q)$ is a lot more complicated because we have first to determine f_0 and then to compute the central charge. Thus the error bar for the central charge would be a lot larger than for the periodic boundary case. We have performed simulations for $q = 8$ and we have obtained $c(8) \simeq 1 - 2$, which is compatible with the result for the same model with periodic boundary conditions.

In conclusion, we have studied the influence of disorder on some q-state Potts models, with $q = 5, 8, 10, 12$ and 256 by measuring the central charge. We have found that the central charge is not a multiple of $\frac{1}{2}$ and thus these models *do not* behave like multiple Ising models for a general q. But when q is a power of 2, we do find a half-integer value for the central charge. For other values of q, we found some new conformal field theories. In addition we found a factorization law such that $c_{q_1 q_2} = c_{q_1} + c_{q_2}$. All these measurements were performed on very small strip width, and thus the precision on which the latter

prediction is made is small. More extensive simulations, and for other values of q should be performed in order to reinforce this result.

This research was supported in part by National Science Foundation under Grant No. PHY94-07194. The author thanks the Institute for Theoretical Physics at Santa Barbara, where this work was started, for its hospitality.

References

[1] Vik. S. Dotsenko and Vl. S. Dotsenko, *Adv. Phys.* **32**, 129 (1983).

[2] A. W. W. Ludwig, *Nucl. Phys.* **B 285**, 97 (1987); *Nucl. Phys.* **B 330**, 639 (1990).

[3] Vl. S. Dotsenko, M. Picco and P. Pujol, *Phys. Lett.* **B 347**, 113 (1995); *Nucl. Phys.* **B 455**, 701 (1995).

[4] V. B. Andreichenko, Vl. S. Dotsenko, W. Selke and J. -S. Wang, *Nucl. Phys.* **B 344**, 531 (1990).

[5] M. Picco, *Phys. Rev.* **B 54**, 14930 (1996).

[6] E. Domany and S. Wiseman, *Phys. Rev.* **E 51**, 3074 (1995); *Phys. Rev.* **E 52**, 3469 (1995).

[7] Y. Imry and M. Wortis, *Phys. Rev.* **B 19**, 3581 (1979).

[8] K. Hui and N. Berker, *Phys. Rev. Lett.* **62**, 2507 (1989).

[9] A. Aizenman and J. Wehr, *Phys. Rev. Lett.* **62**, 2503 (1989).

[10] S. Chen, A. M. Ferrenberg and D. P. Landau, *Phys. Rev. Lett.* **69**, 1213 (1992).

[11] J. L. Cardy, *J. of Phys.* **A 26**, 1897 (1996).

[12] P. Pujol, *Europhys. Lett.* **35**, 283 (1996).

[13] H. Blöte, J. L. Cardy and M. P. Nightingale, *Phys. Rev. Lett.* **56**, 742 (1986).

[14] I. Affleck, *Phys. Rev. Lett.* **56**, 746 (1986).

[15] M. P. Nightingale, in *Finite Size Scaling and Numerical Simulations of Statistical Systems*, edited by V. Privman (World Scientific, Singapore, 1990).

[16] W. Kinzel and E. Domany, *Phys. Rev.* **B 23**, 3421 (1981)

[17] H. Blöte and M. P. Nightingale, *Physica* **112 A**, 405 (1982).

[18] J. L. Cardy, *J. of Phys.* **A 19**, L1093 (1986).

[19] J. M. Thijssen and H. J. F. Knops, *Phys. Rev.* **B 42**, 2438 (1990).

L	q=5	8	10	12	256
2	3.383531 (31)	3.825537 (33)	4.039727 (34)	4.216639 (34)	7.367390 (53)
3	3.283052 (24)	3.694779 (26)	3.894575 (27)	4.059728 (27)	7.015583 (40)
4	3.249701 (21)	3.651447 (22)	3.846513 (23)	4.007812 (24)	6.900199 (34)
5	3.234857 (30)	3.632161 (36)	3.825131 (33)	3.984714 (33)	6.849156 (39)
6	3.227029 (142)	3.622131 (130)	3.813995 (137)	3.972503 (135)	6.821969 (199)

Table 1: Free energy as function of q and L.

L	q=5	8	10	12	256
2-3	1.3817 (4)	1.7981 (5)	1.9960 (5)	2.1577 (5)	4.8377 (8)
3-4	1.3095 (9)	1.703 (1)	1.888 (1)	2.040 (1)	4.533 (2)
4-5	1.260 (3)	1.637 (3)	1.813 (3)	1.961 (3)	4.333 (4)
5-6	1.23 (2)	1.57 (2)	1.74 (3)	1.91 (3)	4.25 (3)

Table 2: Central charge $c_1(q)$ from a fit with Eq. (1).

L	q=2	3	4	5	8	10	12	256
2-4	0.53004	0.85335	1.08010	1.243 (3)	1.614 (3)	1.789 (3)	1.930 (3)	4.251 (4)
3-5	0.49875	0.79846	1.00493	1.189 (8)	1.543 (8)	1.709 (8)	1.847 (8)	4.044 (9)
4-6	0.49507	0.79222	0.99610	1.16 (5)	1.45 (6)	1.60 (7)	1.81 (7)	4.08 (9)

Table 3: Central charge $c_2(q)$ from a fit with Eq. (7).